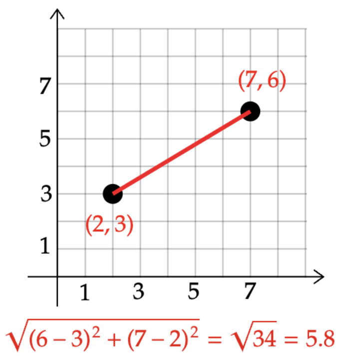
Do Now: Pythagorean Theorem

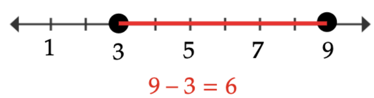
Fatma Moalla made extensive contributions to Finsler spaces. A Finsler space consists of manifolds whose length may be described by what is known as a Minkowski functional. Our goal is simple -- to understand the formula for the Minkowski functional:

$$L(\gamma) = \int_a^b F(\gamma(x), \dot{\gamma}(x))dt$$

Minkowski Functional

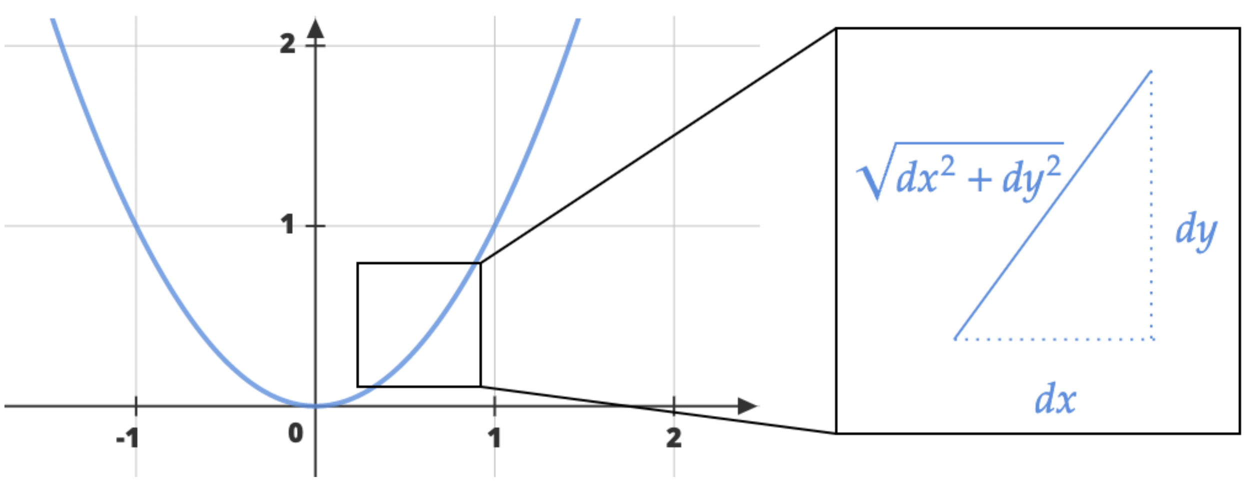
We begin with something simpler: in the two problems below, find the distance between the two points, at first using subtraction, and then the distance formula.





Big Idea: Length of a Curve

We know how to find the length of a line, by using the distance formula to calculate the distance between its endpoints. But how can we find the length of a curve? Well, remember that a curve -- when zoomed in -- is nothing but a line! Thus, we can simply use the pythagorean theorem, but modify it using calculus!



Now, simply integrate the hypotenuse from $x = a$ to $x = b$!

$$\int_a^b \sqrt{dx^2 + dy^2}$$

Let's factor out dx^2:

$$\int_a^b \sqrt{dx^2\left(1 + \frac{dy^2}{dx^2}\right)}$$

We can now pull out the dx^2 from the square root:

$$\int_a^b dx\sqrt{1 + \frac{dy^2}{dx^2}}$$

And now, we are left with the formula for the length of a curve!

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \rightarrow \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

The square root is no more than a function F, and what is it a function of? It is a function of the derivative of the curve $f'(x)$! The derivative may also be written as $\dot{f}(x)$, where the dot on top of the f represents the first derivative. If we make these modifications, we see that the Minkowski functional is quite similar to the length of a curve!

$$\boxed{L(x) = \int_a^b F(\dot{f}(x))dx} \; \textit{Length of Curve}$$

$$\boxed{L(\gamma) = \int_a^b F(\gamma(x), \dot{\gamma}(x))dt} \; \textit{Minkowski Functional}$$

Use your newfound knowledge of the length of a curve to find the length of $y = x^2$ from $x = 1$ to $x = 3$. Note that $\int \csc^3 x\,dx = \frac{-\cot x * \csc x - \ln(|\cot x + \csc x|)}{2} + C$.

$$\int_1^3 \sqrt{1 + (2x)^2}dx = \int_1^3 \sqrt{1 + 4x^2}dx$$

We now factor out 4 from the square root, so as to make the coefficient of x^2 one:

$$\int_1^3 \sqrt{4(\frac{1}{4} + x^2)}dx = \int_1^3 2\sqrt{\frac{1}{4} + x^2}dx$$

Now, make the substitution $x = \frac{1}{2}\cot\theta$ and $dx = -\frac{1}{2}\csc^2\theta\, d\theta$.

$$\int 2\sqrt{\frac{1}{4} + (\frac{1}{2}\cot\theta)^2} * -\frac{1}{2}\csc^2\theta d\theta = \int 2\sqrt{\frac{1}{4} + \frac{1}{4}\cot^2\theta} * -\frac{1}{2}\csc^2\theta d\theta$$

Factoring out $\frac{1}{4}$, we have:

$$\int 2\sqrt{\frac{1}{4}(1 + \cot^2\theta)} * -\frac{1}{2}\csc^2\theta d\theta = \int 2\frac{1}{2}\sqrt{1 + \cot^2\theta} * -\frac{1}{2}\csc^2\theta d\theta$$

Recall that $\sin^2\theta + \cos^2\theta = 1$. Dividing both sides by $\sin^2\theta$, we have $1 + \cot^2\theta = \csc^2\theta$, which results in

$$\int \csc\theta * -\frac{1}{2}\csc^2\theta d\theta = -\frac{1}{2}\int \csc^3\theta\, d\theta$$

We are given this integral in the problem, which gives a final answer of

$$\frac{\cot\theta*\csc\theta+\ln(|\cot\theta+\csc\theta|)}{4}$$